

# On the variances of a spatial unit root model

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## Abstract

Asymptotic properties of the variances of the spatial autoregressive model  $X_{k,\ell} = \alpha X_{k-1,\ell} + \beta X_{k,\ell-1} + \gamma X_{k-1,\ell-1} + \varepsilon_{k,\ell}$  are investigated in the unit root case, that is when the parameters are on the boundary of the domain of stability that forms a tetrahedron in  $[-1, 1]^3$ . The limit of the variance of  $n^{-\varrho} X_{[ns], [nt]}$  is determined, where on the interior of the faces of the domain of stability  $\varrho = 1/4$ , on the edges  $\varrho = 1/2$ , while on the vertices  $\varrho = 1$ .

**Key words:** Spatial autoregressive processes, unit root models.

## 1 Introduction

The analysis of spatial autoregressive models is of interest in many different fields of science such as geography, geology, biology and agriculture. A detailed discussion of these applications is given by Basu and Reinsel (1993) where the authors considered a special case of the so called unilateral autoregressive model having the form

$$X_{k,\ell} = \sum_{i=0}^{p_1} \sum_{j=0}^{p_2} \alpha_{i,j} X_{k-i,\ell-j} + \varepsilon_{k,\ell}, \quad \alpha_{0,0} = 0. \quad (1.1)$$

A particular case of the above model is the doubly geometric spatial autoregressive process

$$X_{k,\ell} = \alpha X_{k-1,\ell} + \beta X_{k,\ell-1} - \alpha\beta X_{k-1,\ell-1} + \varepsilon_{k,\ell},$$

introduced by Martin (1979). This was the first spatial autoregressive model for which unstability has been studied. It is, in fact, the simplest spatial model, since the product structure  $\varphi(x, y) = xy - \alpha x - \beta y + \alpha\beta = (x - \alpha)(y - \beta)$  of its characteristic polynomial ensures that it can be considered as some kind of combination of two autoregressive processes on the line, and several properties can be derived by the analogy of one-dimensional autoregressive processes. This model has been used by Jain (1981) in the study of image processing, by

Martin (1990), Cullis and Gleeson (1991), Basu and Reinsel (1994) in agricultural trials and by Tjøstheim (1981) in digital filtering.

In the stable case when  $|\alpha| < 1$  and  $|\beta| < 1$ , asymptotic normality of several estimators  $(\hat{\alpha}_{m,n}, \hat{\beta}_{m,n})$  of  $(\alpha, \beta)$  based on the observations  $\{X_{k,\ell} : 1 \leq k \leq m \text{ and } 1 \leq \ell \leq n\}$  has been shown (e.g. Tjøstheim (1978, 1983) or Basu and Reinsel (1992, 1993)), namely,

$$\sqrt{mn} \begin{pmatrix} \hat{\alpha}_{m,n} - \alpha \\ \hat{\beta}_{m,n} - \beta \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma_{\alpha,\beta})$$

as  $m, n \rightarrow \infty$  with  $m/n \rightarrow \text{constant} > 0$  with some covariance matrix  $\Sigma_{\alpha,\beta}$ .

In the unstable case when  $\alpha = \beta = 1$ , in contrast to the classical first order autoregressive time series model, where the appropriately normed least squares estimator (LSE) of the autoregressive parameter converges to a fraction of functionals of the standard Brownian motion (see e.g. Phillips (1987) or Chan and Wei (1987)), the sequence of Gauss–Newton estimators  $(\hat{\alpha}_{n,n}, \hat{\beta}_{n,n})$  of  $(\alpha, \beta)$  has been shown to be asymptotically normal (see Bhattacharyya *et al.* (1996) and Bhattacharyya *et al.* (1997)). In the unstable case  $\alpha = 1$ ,  $|\beta| < 1$  the LSE turns out to be asymptotically normal again (Bhattacharyya *et al.*, 1996).

Baran *et al.* (2004) discussed a special case of the model (1.1), namely, when  $p_1 = p_2 = 1$ ,  $\alpha_{0,1} = \alpha_{1,0} =: \alpha$  and  $\alpha_{1,1} = 0$ , which is the simplest spatial model, that can not be reduced somehow to autoregressive models on the line. This model is stable in case  $|\alpha| < 1/2$  (see e.g. Whittle (1954), Besag (1972) or Basu and Reinsel (1993)), and unstable if  $|\alpha| = 1/2$ . In Baran *et al.* (2004) the asymptotic normality of the LSE of the unknown parameter  $\alpha$  is proved both in stable and unstable cases. The case  $p_1 = p_2 = 1$ ,  $\alpha_{1,0} =: \alpha$ ,  $\alpha_{0,1} =: \beta$  and  $\alpha_{1,1} = 0$  was studied by Paulauskas (2007) and Baran *et al.* (2007). This model is stable in case  $|\alpha| + |\beta| < 1$  and unstable if  $|\alpha| + |\beta| = 1$  (Basu and Reinsel, 1993). Paulauskas (2007) determined the exact asymptotic behaviour of the variances of the process, while Baran *et al.* (2007) proved the asymptotic normality of the LSE of the parameters both in stable and unstable cases.

In the present paper we study the asymptotic properties of a more complicated special case of the model (1.1) with  $p_1 = p_2 = 1$ ,  $\alpha_{1,0} =: \alpha$ ,  $\alpha_{0,1} =: \beta$  and  $\alpha_{1,1} =: \gamma$ . Our aim is to clarify the asymptotic behaviour of the variances. The asymptotic results on the variances (and covariances) help in finding the asymptotic properties of various estimators of the autoregressive parameters (see e.g. Baran *et al.* (2004, 2007)).

We consider the spatial autoregressive process  $\{X_{k,\ell} : k, \ell \in \mathbb{Z}, k, \ell \geq 0\}$  is defined as

$$\begin{cases} X_{k,\ell} = \alpha X_{k-1,\ell} + \beta X_{k,\ell-1} + \gamma X_{k-1,\ell-1} + \varepsilon_{k,\ell}, & \text{for } k, \ell \geq 1, \\ X_{k,0} = X_{0,\ell} = 0, & \text{for } k, \ell \geq 0. \end{cases} \quad (1.2)$$

The model is stable if  $|\alpha| < 1$ ,  $|\beta| < 1$  and  $|\gamma| < 1$ ,  $|1 + \alpha^2 - \beta^2 - \gamma^2| > 2|\alpha + \beta\gamma|$  and  $1 - \beta^2 > |\alpha + \beta\gamma|$ , and unstable on the boundary of this domain (Basu and Reinsel, 1993) (see

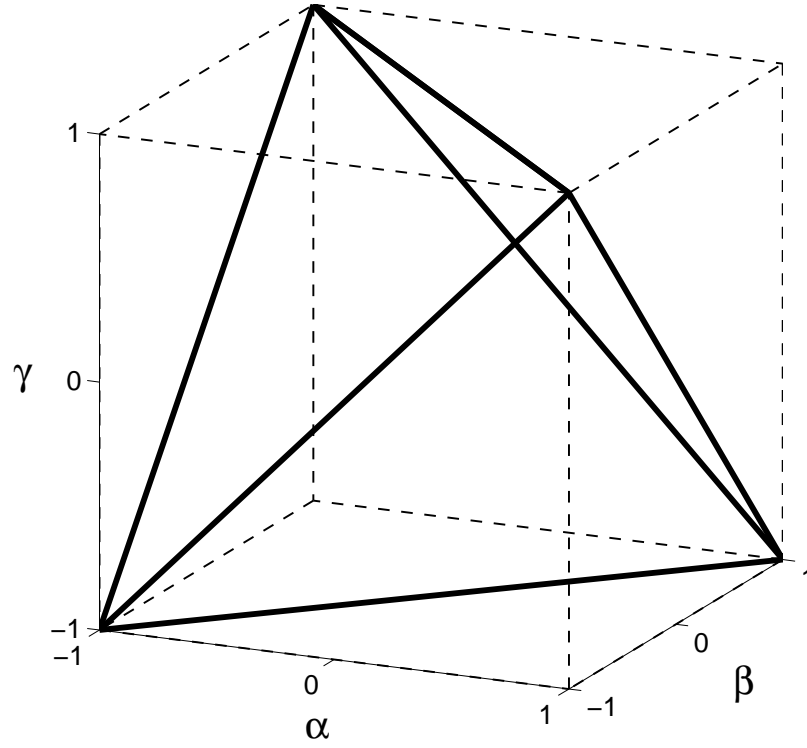


Figure 1: The domain of stability of model (1.2).

Figure 1). Short calculation shows that condition of stability means that  $|\alpha| < 1$ ,  $|\beta| < 1$  and  $|\gamma| < 1$ , and inequalities

$$\alpha - \beta - \gamma < 1, \quad -\alpha + \beta - \gamma < 1, \quad -\alpha - \beta + \gamma < 1, \quad \alpha + \beta + \gamma < 1$$

hold. Obviously, in case  $\alpha\beta\gamma \geq 0$  the above set of conditions reduces to  $|\alpha| + |\beta| + |\gamma| < 1$ . If the model is not stable, one can distinguish three cases:

CASE A. The parameters are in the interior of the faces of the boundary of the domain of stability, i.e.  $|\alpha| < 1$ ,  $|\beta| < 1$ ,  $|\gamma| < 1$  and one of the following equations is fulfilled

$$\alpha - \beta - \gamma = 1; \quad -\alpha + \beta - \gamma = 1; \quad -\alpha - \beta + \gamma = 1; \quad \alpha + \beta + \gamma = 1. \quad (1.3)$$

We remark that in case  $\alpha\beta\gamma \geq 0$  the set of equations (1.3) is equivalent to  $|\alpha| + |\beta| + |\gamma| = 1$ , while in case  $\alpha\beta\gamma < 0$  to

$$|\alpha| + |\beta| - |\gamma| = 1 \quad \text{or} \quad |\alpha| - |\beta| + |\gamma| = 1 \quad \text{or} \quad -|\alpha| + |\beta| + |\gamma| = 1.$$

CASE B. The parameters are in the interior of the edges of the boundary of the domain of stability, i.e.  $\alpha\beta\gamma \leq 0$  and one of the following equations is fulfilled

$$|\alpha| = 1 \quad \text{and} \quad |\beta| = |\gamma| < 1; \quad |\beta| = 1 \quad \text{and} \quad |\alpha| = |\gamma| < 1; \quad |\gamma| = 1 \quad \text{and} \quad |\alpha| = |\beta| < 1.$$

Observe that in each of the above three cases exactly two of the defining equations (1.3) of Case A are satisfied. In this way Case B can be considered as an extension of Case A to the situation when  $\alpha\beta\gamma \leq 0$  and one of the parameters equals  $\pm 1$ , while the other two parameters have absolute values less than one.

Further, observe that in the first two cases  $\gamma = -\alpha\beta$ , so we obtain spacial cases of the doubly geometric model. If  $|\alpha| = 1$ ,  $|\beta| = |\gamma| \leq 1$  then for  $k \in \mathbb{N}$  the difference  $\Delta_{1,\alpha}X_{k,\ell} := X_{k,\ell} - \alpha X_{k-1,\ell}$  is a classical AR(1) process, i.e.  $\Delta_{1,\alpha}X_{k,\ell} = \beta\Delta_{1,\alpha}X_{k,\ell-1} + \varepsilon_{k,\ell}$ . Similarly, if  $|\beta| = 1$ ,  $|\alpha| = |\gamma| \leq 1$  then  $\Delta_{2,\beta}X_{k,\ell} = \alpha\Delta_{2,\beta}X_{k-1,\ell} + \varepsilon_{k,\ell}$ , where  $\Delta_{2,\beta}X_{k,\ell} := X_{k,\ell} - \beta X_{k,\ell-1}$ ,  $\ell \in \mathbb{N}$ .

CASE C. The parameters are in the vertices of the boundary of the domain of stability, i.e.  $\alpha\beta\gamma = -1$  and  $|\alpha| = |\beta| = |\gamma| = 1$ .

**Theorem 1.1** *Let  $\{\varepsilon_{k,\ell} : k, \ell \in \mathbb{N}\}$  be independent random variables with  $\mathbb{E} \varepsilon_{k,\ell} = 0$  and  $\text{Var} \varepsilon_{k,\ell} = 1$ . Assume that model (1.2) holds and for  $n \in \mathbb{N}$  consider the piecewise constant random field*

$$Y^{(n)}(s, t) := X_{[ns], [nt]}, \quad s, t \in \mathbb{R}, \quad s, t \geq 0.$$

*If  $|\alpha| < 1$ ,  $|\beta| < 1$  and  $|\gamma| < 1$ ,  $|1 + \alpha^2 - \beta^2 - \gamma^2| > 2|\alpha + \beta\gamma|$  and  $1 - \beta^2 > |\alpha + \beta\gamma|$  then*

$$\lim_{n \rightarrow \infty} \text{Var}(Y^{(n)}(s, t)) = \sigma_{\alpha, \beta, \gamma}^2 := ((1 + \alpha + \beta - \gamma)(1 + \alpha - \beta + \gamma)(1 - \alpha + \beta + \gamma)(1 - \alpha - \beta - \gamma))^{-1/2}.$$

*If  $|\alpha| < 1$ ,  $|\beta| < 1$  and  $|\gamma| < 1$  and in case  $\alpha\beta\gamma \geq 0$  equation  $|\alpha| + |\beta| + |\gamma| = 1$ , while in case  $\alpha\beta\gamma < 0$  equation  $|\alpha| + |\beta| - |\gamma| = 1$  holds then*

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} \text{Var}(Y^{(n)}(s, t)) = \frac{((1 - |\alpha|)s)^{1/2} \wedge ((1 - |\beta|)t)^{1/2}}{\pi^{1/2}(|\alpha| + |\beta|)^{1/2}(1 - |\alpha|)(1 - |\beta|)}.$$

*If  $\alpha\beta\gamma \leq 0$  and  $|\alpha| = 1$ ,  $|\beta| = |\gamma| < 1$  or  $|\beta| = 1$ ,  $|\alpha| = |\gamma| < 1$  then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Var}(Y^{(n)}(s, t)) = \frac{s}{1 - \gamma^2} \quad \text{or} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \text{Var}(Y^{(n)}(s, t)) = \frac{t}{1 - \gamma^2},$$

*respectively.*

*If  $\alpha\beta\gamma = -1$  and  $|\alpha| = |\beta| = |\gamma| = 1$  then*

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \text{Var}(Y^{(n)}(s, t)) = st.$$

Observe that in the last case the limit of the variances is exactly the variance of the standard Wiener sheet. This result is quite natural, as e.g. for  $\alpha = \beta = -\gamma = 1$  model equation (1.2) reduces to  $\Delta_{1,1}\Delta_{2,1}X_{k,\ell} = \varepsilon_{k,\ell}$ .

We remark that results given Theorem 1.1 do not cover all possible locations of the parameters on the boundary of the domain of stability. Some results on the missing cases can be found in Section 4.

The aim of the following discussion is to show that it suffices to prove Theorem 1.1 for  $\alpha \geq 0$ ,  $\beta \geq 0$  and  $\gamma \geq 0$  if  $\alpha\beta\gamma \geq 0$  and for  $\alpha > 0$ ,  $\beta > 0$  and  $\gamma < 0$  if  $\alpha\beta\gamma < 0$ . First we note that direct calculations imply

$$X_{k,\ell} = \sum_{i=1}^k \sum_{j=1}^{\ell} G(k-i, \ell-j; \alpha, \beta, \gamma) \varepsilon_{i,j} \quad (1.4)$$

$$= \sum_{i=1}^k \sum_{j=1}^{\ell} \binom{k+\ell-i-j}{\ell-j} \alpha^{k-i} \beta^{\ell-j} F\left(i-k, j-\ell; i+j-k-\ell; -\frac{\gamma}{\alpha\beta}\right) \varepsilon_{i,j}, \quad (1.5)$$

$k, \ell \geq 1$ , where (1.5) holds only for  $\alpha\beta \neq 0$ ,

$$G(m, n; \alpha, \beta, \gamma) := \sum_{r=0}^{m \wedge n} \frac{(m+n-r)!}{(m-r)!(n-r)!r!} \alpha^{m-r} \beta^{n-r} \gamma^r, \quad m, n \in \mathbb{N} \cup \{0\},$$

and  $F(-n, b; c; z)$  is the Gauss hypergeometric function defined by

$$F(-n, b; c; z) := \sum_{r=0}^n \frac{(-n)_r (b)_r}{(c)_r r!} z^r, \quad n \in \mathbb{N}, \quad b, c, z \in \mathbb{C},$$

and  $(a)_r := a(a+1)\dots(a+r-1)$  (for the definition in more general cases see e.g. Bateman and Erdélyi (1953)).

Observe that as for  $m, n \in \mathbb{N}$  we have  $F(-n, -m; -n-m; 1) = \binom{m+n}{n}^{-1}$  and  $F(-n, -m; -n-m; 0) = 1$ , moving average representations of the doubly geometric model of Martin (1979) and of the spatial models studied by Paulauskas (2007) and Baran *et al.* (2004, 2007), respectively, are special forms of (1.5).

Now, put  $\tilde{\varepsilon}_{k,\ell} := (-1)^{k+\ell} \varepsilon_{k,\ell}$  for  $k, \ell \in \mathbb{N}$ . Then  $\{\tilde{\varepsilon}_{k,\ell} : k, \ell \in \mathbb{N}\}$  are independent random variables with  $\mathbb{E} \tilde{\varepsilon}_{k,\ell} = 0$  and  $\text{Var} \tilde{\varepsilon}_{k,\ell} = 1$ . Consider the process  $\{\tilde{X}_{k,\ell} : k, \ell \in \mathbb{Z}, k, \ell \geq 0\}$  defined as

$$\begin{cases} \tilde{X}_{k,\ell} = -\alpha \tilde{X}_{k-1,\ell} - \beta \tilde{X}_{k,\ell-1} + \gamma \tilde{X}_{k-1,\ell-1} + \tilde{\varepsilon}_{k,\ell}, & \text{for } k, \ell \geq 1, \\ \tilde{X}_{k,0} = \tilde{X}_{0,\ell} = 0, & \text{for } k, \ell \geq 0. \end{cases}$$

Then by representation (1.4) for  $k, \ell \in \mathbb{N}$  we have

$$\begin{aligned} \tilde{X}_{k,\ell} &= \sum_{i=1}^k \sum_{j=1}^{\ell} G(k-i, \ell-j; -\alpha, -\beta, \gamma) \tilde{\varepsilon}_{i,j} \\ &= \sum_{i=1}^k \sum_{j=1}^{\ell} (-1)^{k+\ell-i-j} G(k-i, \ell-j; \alpha, \beta, \gamma) \tilde{\varepsilon}_{i,j} = (-1)^{k+\ell} X_{k,\ell}, \end{aligned}$$

hence  $\text{Var} \tilde{X}_{k,\ell} = \text{Var} X_{k,\ell}$ .

Next, put  $\hat{\varepsilon}_{k,\ell} := (-1)^k \varepsilon_{k,\ell}$  for  $k, \ell \in \mathbb{N}$ . Then  $\{\hat{\varepsilon}_{k,\ell} : k, \ell \in \mathbb{N}\}$  are again independent random variables with  $\mathbb{E} \hat{\varepsilon}_{k,\ell} = 0$  and  $\text{Var} \hat{\varepsilon}_{k,\ell} = 1$ . Consider the process  $\{\hat{X}_{k,\ell} : k, \ell \in \mathbb{Z}, k, \ell \geq 0\}$  defined as

$$\begin{cases} \hat{X}_{k,\ell} = -\alpha \hat{X}_{k-1,\ell} + \beta \hat{X}_{k,\ell-1} - \gamma \hat{X}_{k-1,\ell-1} + \hat{\varepsilon}_{k,\ell}, & \text{for } k, \ell \geq 1, \\ \hat{X}_{k,0} = \hat{X}_{0,\ell} = 0, & \text{for } k, \ell \geq 0. \end{cases}$$

Then by representation (1.4) for  $k, \ell \in \mathbb{N}$  we have

$$\begin{aligned} \hat{X}_{k,\ell} &= \sum_{i=1}^k \sum_{j=1}^{\ell} G(k-i, \ell-j; -\alpha, \beta, -\gamma) \hat{\varepsilon}_{i,j} \\ &= \sum_{i=1}^k \sum_{j=1}^{\ell} (-1)^{k-i} G(k-i, \ell-j; \alpha, \beta, \gamma) \hat{\varepsilon}_{i,j} = (-1)^k X_{k,\ell}, \end{aligned}$$

hence  $\text{Var} \hat{X}_{k,\ell} = \text{Var} X_{k,\ell}$ .

In a similar way we have  $\bar{X}_{k,\ell} = (-1)^\ell X_{k,\ell}$ , so  $\text{Var} \bar{X}_{k,\ell} = \text{Var} X_{k,\ell}$ , where  $\{\bar{X}_{k,\ell} : k, \ell \in \mathbb{Z}, k, \ell \geq 0\}$  is defined as

$$\begin{cases} \bar{X}_{k,\ell} = \alpha \bar{X}_{k-1,\ell} - \beta \bar{X}_{k,\ell-1} - \gamma \bar{X}_{k-1,\ell-1} + \bar{\varepsilon}_{k,\ell}, & \text{for } k, \ell \geq 1, \\ \bar{X}_{k,0} = \bar{X}_{0,\ell} = 1. & \text{for } k, \ell \geq 0, \end{cases}$$

with  $\bar{\varepsilon}_{k,\ell} := (-1)^\ell \varepsilon_{k,\ell}$ .

## 2 Upper bounds for the covariances

By representations (1.4) and (1.5) we obtain that for  $k_1, \ell_1, k_2, \ell_2 \in \mathbb{N}$  and  $\alpha, \beta, \gamma \in \mathbb{R}$  we have

$$\text{Cov}(X_{k_1, \ell_1}, X_{k_2, \ell_2}) = \sum_{i=1}^{k_1 \wedge k_2} \sum_{j=1}^{\ell_1 \wedge \ell_2} G(k_1 - i, \ell_1 - j; \alpha, \beta, \gamma) G(k_2 - i, \ell_2 - j; \alpha, \beta, \gamma) \quad (2.1)$$

$$\begin{aligned} &= \sum_{i=1}^{k_1 \wedge k_2} \sum_{j=1}^{\ell_1 \wedge \ell_2} \binom{k_1 + \ell_1 - i - j}{\ell_1 - j} \binom{k_2 + \ell_2 - i - j}{\ell_2 - j} \alpha^{k_1 + k_2 - 2i} \beta^{\ell_1 + \ell_2 - 2j} \\ &\quad \times F\left(i - k_1, j - \ell_1; i + j - k_1 - \ell_1; -\frac{\gamma}{\alpha\beta}\right) F\left(i - k_2, j - \ell_2; i + j - k_2 - \ell_2; -\frac{\gamma}{\alpha\beta}\right), \end{aligned} \quad (2.2)$$

where  $x \wedge y := \min\{x, y\}$ ,  $x, y \in \mathbb{R}$ , and (2.2) holds only for  $\alpha\beta \neq 0$ .

To obtain a more convenient form of the covariances we prove the following Lemma.

**Lemma 2.1** *Let  $n, m$  be nonnegative integers and let  $\alpha, \beta, \gamma \in \mathbb{R}$  such that  $0 \leq \alpha, \beta < 1$  and  $\alpha\beta + \gamma \geq 0$ . Then*

$$G(m, n; \alpha, \beta, \gamma) = \left( \frac{\alpha + \gamma}{1 - \beta} \right)^m \mathbb{P} \left( \xi_n^{(\beta)} + \eta_m^{\left( \frac{\alpha\beta + \gamma}{\alpha + \gamma} \right)} = n \right) = \left( \frac{\beta + \gamma}{1 - \alpha} \right)^n \mathbb{P} \left( \xi_m^{(\alpha)} + \eta_n^{\left( \frac{\alpha\beta + \gamma}{\beta + \gamma} \right)} = m \right),$$

where  $\xi_n^{(\nu)}$  and  $\eta_m^{(\mu)}$ ,  $0 \leq \mu, \nu \leq 1$ , are independent binomial random variables with parameters  $(n, \nu)$  and  $(m, \mu)$ , respectively, if  $m, n \in \mathbb{N}$ , and  $\xi_0^{(\nu)} = \eta_0^{(\mu)} := 0$ .

**Proof** As in cases  $\alpha\beta\gamma = 0$  or  $\alpha\beta + \gamma = 0$  the statement of the Lemma holds trivially, we may assume  $\alpha\beta\gamma \neq 0$  and  $\alpha\beta + \gamma > 0$ . Let  $0 < n \leq m$ .

$$\begin{aligned} G(m, n; \alpha, \beta, \gamma) &= \sum_{r=0}^{m \wedge n} \frac{(m+n-r)!}{(m-r)!(n-r)!r!} \alpha^{m-r} \beta^{n-r} \gamma^r = \frac{(\beta + |\gamma|)^n (\text{sign}(\gamma))^m}{n!} \\ &\times \sum_{r=0}^n \frac{(n+m-r)!}{(m-r)!} \alpha_\gamma^{m-r} \mathbb{P}(\xi_n^{(\nu)} = n-r) = \frac{(\beta + |\gamma|)^n (\text{sign}(\gamma))^m}{n!} \frac{d^n (\alpha_\gamma^m \mathcal{G}_n^{(\nu)}(\alpha_\gamma))}{d\alpha_\gamma^n}, \end{aligned}$$

where  $\nu := \frac{\beta}{\beta + |\gamma|}$ ,  $\alpha_\gamma := \text{sign}(\gamma) \alpha$  and  $\mathcal{G}_n^{(\nu)}(x) := (\nu x + (1-\nu))^n$  is the generating function of  $\xi_n^{(\nu)}$ . From the other hand

$$\frac{d^n (\alpha_\gamma^m \mathcal{G}_n^{(\nu)}(\alpha_\gamma))}{d\alpha_\gamma^n} = n! \sum_{r=0}^n \binom{n}{r} \binom{m}{n-r} \nu^r (\nu \alpha_\gamma + (1-\nu))^{n-r} \alpha_\gamma^{m-(n-r)},$$

and as  $\alpha\beta + \gamma < \alpha + \gamma$  we obtain

$$\begin{aligned} G(m, n; \alpha, \beta, \gamma) &= \sum_{r=0}^n \binom{n}{r} \binom{m}{n-r} \beta^r (\alpha\beta + \gamma)^{n-r} \alpha^{m-n+r} \\ &= \left( \frac{\alpha + \gamma}{1 - \beta} \right)^m \sum_{r=0}^n \binom{n}{r} \beta^r (1 - \beta)^{n-r} \binom{m}{n-r} \left( \frac{\alpha\beta + \gamma}{\alpha + \gamma} \right)^{n-r} \left( \frac{\alpha(1 - \beta)}{\alpha + \gamma} \right)^{m-n+r} \\ &= \left( \frac{\alpha + \gamma}{1 - \beta} \right)^m \sum_{r=0}^n \mathbb{P}(\xi_n^{(\beta)} = r) \mathbb{P} \left( \eta_m^{\left( \frac{\alpha\beta + \gamma}{\alpha + \gamma} \right)} = n-r \right) = \left( \frac{\alpha + \gamma}{1 - \beta} \right)^m \mathbb{P} \left( \xi_n^{(\beta)} + \eta_m^{\left( \frac{\alpha\beta + \gamma}{\alpha + \gamma} \right)} = n \right). \end{aligned} \tag{2.3}$$

Moreover,

$$\begin{aligned} \sum_{r=0}^n \binom{n}{r} \binom{m}{n-r} \beta^r (\alpha\beta + \gamma)^{n-r} \alpha^{m-n+r} &= \sum_{r=m-n}^m \binom{m}{r} \binom{n}{m-r} \alpha^r (\alpha\beta + \gamma)^{m-r} \beta^{n-m+r} \\ &= \left( \frac{\beta + \gamma}{1 - \alpha} \right)^n \sum_{r=m-n}^m \mathbb{P}(\xi_m^{(\alpha)} = r) \mathbb{P} \left( \eta_n^{\left( \frac{\alpha\beta + \gamma}{\beta + \gamma} \right)} = m-r \right) = \left( \frac{\beta + \gamma}{1 - \alpha} \right)^n \mathbb{P} \left( \xi_m^{(\alpha)} + \eta_n^{\left( \frac{\alpha\beta + \gamma}{\beta + \gamma} \right)} = m \right) \end{aligned}$$

that together with (2.3) implies the statement of the lemma. Case  $n > m$  can be handled in a similar way.  $\square$

**Corollary 2.2** *If  $0 \leq \alpha, \beta < 1$  and  $\alpha + \beta + \gamma = 1$  then*

$$G(m, n; \alpha, \beta, \gamma) = \mathbb{P}\left(\xi_m^{(\alpha)} + \eta_n^{(1-\beta)} = m\right) = \mathbb{P}\left(\xi_n^{(\beta)} + \eta_m^{(1-\alpha)} = n\right).$$

The following lemma is a natural generalization of Theorem 2.4 of Baran *et al.* (2007).

**Lemma 2.3** *Let  $k, \ell \in \mathbb{N}$ , let  $0 < \mu, \nu < 1$  be real numbers and let  $\xi_k^{(\nu)}$  and  $\eta_\ell^{(\mu)}$  be independent binomial random variables with parameters  $(k, \nu)$  and  $(\ell, \mu)$ , respectively. Further, let  $S_{k,\ell} := \xi_k^{(\nu)} + \eta_\ell^{(\mu)}$  and let*

$$m_{k,\ell} := \mathbb{E}S_{k,\ell}, \quad b_{k,\ell} := \text{Var}S_{k,\ell}, \quad x_{j,k,\ell} := (j - m_{k,\ell})/\sqrt{b_{k,\ell}}.$$

*Then for all  $k, \ell \in \mathbb{N}$  and  $j \in \{0, 1, \dots, k + \ell\}$ , we have*

$$\left| \mathbb{P}(S_{k,\ell} = j) - \frac{1}{\sqrt{2\pi b_{k,\ell}}} \exp(-x_{j,k,\ell}^2/2) \right| \leq \frac{C(\mu, \nu)}{b_{k,\ell}},$$

*where  $C(\mu, \nu) > 0$  is a constant depending only on  $\mu$  and  $\nu$  (and not depending on  $k, \ell, j$ ).*

**Theorem 2.4** *If  $|\alpha| + |\beta| + |\gamma| < 1$  then*

$$|\text{Cov}(X_{k_1, \ell_1}, X_{k_2, \ell_2})| \leq \frac{(|\alpha| + |\beta| + |\gamma|)^{(|k_1 - k_2| + |\ell_1 - \ell_2|)/2}}{(1 - (|\alpha| + |\beta| + |\gamma|))^2}.$$

*If  $|\alpha| < 1$ ,  $|\beta| < 1$  and  $|\gamma| < 1$  and in case  $\alpha\beta\gamma \geq 0$  equation  $|\alpha| + |\beta| + |\gamma| = 1$ , while in case  $\alpha\beta\gamma < 0$  equation  $|\alpha| + |\beta| - |\gamma| = 1$  holds then*

$$|\text{Cov}(X_{k_1, \ell_1}, X_{k_2, \ell_2})| \leq C(\alpha, \beta) \sqrt{k_1 + \ell_1 + k_2 + \ell_2}$$

*with some constant  $C(\alpha, \beta) > 0$ .*

*If  $\alpha\beta\gamma \leq 0$  and  $|\alpha| = 1$ ,  $|\beta| = |\gamma| < 1$  or  $|\beta| = 1$ ,  $|\alpha| = |\gamma| < 1$  then*

$$|\text{Cov}(X_{k_1, \ell_1}, X_{k_2, \ell_2})| \leq (k_1 \wedge k_2) \frac{|\gamma|^{|\ell_1 - \ell_2|}}{1 - \gamma^2} \quad \text{or} \quad |\text{Cov}(X_{k_1, \ell_1}, X_{k_2, \ell_2})| \leq (\ell_1 \wedge \ell_2) \frac{|\gamma|^{|k_1 - k_2|}}{1 - \gamma^2},$$

*respectively.*

*If  $\alpha\beta\gamma = -1$  and  $|\alpha| = |\beta| = |\gamma| = 1$  then*

$$\text{Cov}(X_{k_1, \ell_1}, X_{k_2, \ell_2}) = (k_1 \wedge k_2)(\ell_1 \wedge \ell_2) \alpha^{|k_1 - k_2|} \beta^{|\ell_1 - \ell_2|}.$$

**Proof.** Let  $|\alpha| + |\beta| + |\gamma| < 1$ . Lemma 2.1 and (2.1) imply

$$\begin{aligned} |\text{Cov}(X_{k_1, \ell_1}, X_{k_2, \ell_2})| &\leq \sum_{i=1}^{k_1 \wedge k_2} \sum_{j=1}^{\ell_1 \wedge \ell_2} \left( \frac{|\alpha| + |\gamma|}{1 - |\beta|} \right)^{(k_1 + k_2)/2 - i} \left( \frac{|\beta| + |\gamma|}{1 - |\alpha|} \right)^{(\ell_1 + \ell_2)/2 - j} \\ &\leq \left( \frac{|\alpha| + |\gamma|}{1 - |\beta|} \right)^{|k_1 - k_2|/2} \left( \frac{|\beta| + |\gamma|}{1 - |\alpha|} \right)^{|\ell_1 - \ell_2|/2} \frac{(1 - |\beta|)(1 - |\alpha|)}{(1 - (|\alpha| + |\beta| + |\gamma|))^2}. \end{aligned}$$



Hence, as

$$\frac{|\alpha| + |\gamma|}{1 - |\beta|} \leq |\alpha| + |\beta| + |\gamma| \quad \text{and} \quad \frac{|\beta| + |\gamma|}{1 - |\alpha|} \leq |\alpha| + |\beta| + |\gamma|$$

hold, we obtain the first statement of the theorem.

Now, let  $|\alpha| < 1$ ,  $|\beta| < 1$ ,  $|\gamma| < 1$  and assume that in case  $\alpha\beta\gamma \geq 0$  equation  $|\alpha| + |\beta| + |\gamma| = 1$ , while in case  $\alpha\beta\gamma < 0$  equation  $|\alpha| + |\beta| - |\gamma| = 1$  holds. From the arguments of the Introduction follows that it suffices to consider the case  $0 \leq \alpha, \beta < 1$ ,  $|\gamma| < 1$ , and  $\alpha + \beta + \gamma = 1$ . Corollary 2.2 and (2.1) imply

$$|\text{Cov}(X_{k_1, \ell_1}, X_{k_2, \ell_2})| \leq \sum_{i=1}^{k_1 \wedge k_2} \sum_{j=1}^{\ell_1 \wedge \ell_2} \mathbb{P}\left(\xi_{k_1-i}^{(\alpha)} + \eta_{\ell_1-j}^{(1-\beta)} = k_1 - i\right) \mathbb{P}\left(\xi_{k_2-i}^{(\alpha)} + \eta_{\ell_2-j}^{(1-\beta)} = k_2 - i\right).$$

Assume first  $k_1 \leq k_2$  and  $\ell_1 \leq \ell_2$  or  $k_1 > k_2$  and  $\ell_1 > \ell_2$ . In this case using the notations and results of Lemma 2.3 we have

$$\begin{aligned} |\text{Cov}(X_{k_1, \ell_1}, X_{k_2, \ell_2})| &\leq \sum_{i=0}^{k_1 \wedge k_2 - 1} \sum_{j=0}^{\ell_1 \wedge \ell_2 - 1} \mathbb{P}\left(\xi_{|k_1-k_2|+i}^{(\alpha)} + \eta_{|\ell_1-\ell_2|+j}^{(1-\beta)} = |k_1-k_2|+i\right) \mathbb{P}\left(\xi_i^{(\alpha)} + \eta_j^{(1-\beta)} = i\right) \\ &\leq \sum_{i=2}^{k_1 \wedge k_2 - 1} \sum_{j=2}^{\ell_1 \wedge \ell_2 - 1} \frac{1}{2\pi \sqrt{b_{|k_1-k_2|+i, |\ell_1-\ell_2|+j}} \sqrt{b_{i,j}}} \exp\left(-\frac{x_{|k_1-k_2|+i, |k_1-k_2|+i, |\ell_1-\ell_2|+j}^2}{2} - \frac{x_{i,j}^2}{2}\right), \\ &\quad + C(\alpha, \beta) \left( \sum_{i=2}^{k_1 \wedge k_2 - 1} \frac{1}{i} + \sum_{j=2}^{\ell_1 \wedge \ell_2 - 1} \frac{1}{j} + \sum_{i=2}^{k_1 \wedge k_2 - 1} \sum_{j=2}^{\ell_1 \wedge \ell_2 - 1} b_{i,j}^{-3/2} \right) + 4, \end{aligned}$$

where  $C(\alpha, \beta)$  is a positive constant and

$$b_{k,\ell} := \alpha(1-\alpha)k + \beta(1-\beta)\ell \quad \text{and} \quad x_{k,k,\ell} := a_{k,\ell} / \sqrt{b_{k,\ell}} \quad \text{with} \quad a_{k,\ell} := (1-\alpha)k - (1-\beta)\ell.$$

Obviously,

$$\sum_{i=2}^{k_1 \wedge k_2 - 1} \frac{1}{i} \leq 2\sqrt{k_1 \wedge k_2} \leq 2\sqrt{k_1 + k_2} \quad \text{and} \quad \sum_{j=2}^{\ell_1 \wedge \ell_2 - 1} \frac{1}{j} \leq 2\sqrt{\ell_1 \wedge \ell_2} \leq 2\sqrt{\ell_1 + \ell_2}.$$

Further, we have

$$\begin{aligned} \sum_{i=2}^{k_1 \wedge k_2 - 1} \sum_{j=2}^{\ell_1 \wedge \ell_2 - 1} b_{i,j}^{-3/2} &= \sum_{i=2}^{k_1 \wedge k_2 - 1} \sum_{j=2}^{\ell_1 \wedge \ell_2 - 1} (\alpha(1-\alpha)i + \beta(1-\beta)j)^{-3/2} \\ &\leq \sum_{i=2}^{k_1 \wedge k_2 - 1} \int_1^{\ell_1 \wedge \ell_2 - 1} (\alpha(1-\alpha)i + \beta(1-\beta)t)^{-3/2} dt \leq \frac{2}{\beta(1-\beta)} \sum_{i=2}^{k_1 \wedge k_2 - 1} (\alpha(1-\alpha)i)^{-1/2} \\ &\leq \frac{2}{\beta(1-\beta)} \int_1^{k_1 \wedge k_2 - 1} (\alpha(1-\alpha)s)^{-1/2} ds \leq \frac{4(\alpha(1-\alpha))^{1/2}}{\alpha\beta(1-\alpha)(1-\beta)} \sqrt{k_1 \wedge k_2}. \end{aligned}$$

Hence,

$$|\text{Cov}(X_{k_1, \ell_1}, X_{k_2, \ell_2})| \leq C(\alpha, \beta) \sqrt{k_1 + \ell_1 + k_2 + \ell_2} + 4H_{\alpha, \beta}(k_1, \ell_1, k_2, \ell_2), \quad (2.4)$$

with

$$\begin{aligned} H_{\alpha, \beta}(k_1, \ell_1, k_2, \ell_2) \\ := \int_1^{k_1 \wedge k_2} \int_1^{\ell_1 \wedge \ell_2} \frac{1}{2\pi \sqrt{b_{|k_1 - k_2| + s, |\ell_1 - \ell_2| + t}} \sqrt{b_{s, t}}} \exp\left(-\frac{x_{|k_1 - k_2| + s, |k_1 - k_2| + s, |\ell_1 - \ell_2| + t}^2}{2} - \frac{x_{s, s, t}^2}{2}\right) dt ds. \end{aligned}$$

It is easy to see that

$$\begin{aligned} H_{\alpha, \beta}(k_1, \ell_1, k_2, \ell_2) &\leq \frac{1}{(\alpha + \beta)(1 - \alpha)(1 - \beta)} \\ &\times \int_{b_{1,1}}^{b_{k_1 \wedge k_2, \ell_1 \wedge \ell_2}} \int_{a_{1, \ell_1 \wedge \ell_2}}^{a_{k_1 \wedge k_2, 1}} \frac{1}{2\pi \sqrt{(b_{|k_1 - k_2|, |\ell_1 - \ell_2|} + u)u}} \exp\left(-\frac{(a_{|k_1 - k_2|, |\ell_1 - \ell_2|} + v)^2}{2(b_{|k_1 - k_2|, |\ell_1 - \ell_2|} + u)} - \frac{v^2}{2u}\right) dv du. \end{aligned}$$

Now, for some real constants  $a < b$  and  $q, \varrho$  we have

$$\begin{aligned} \int_a^b \exp\left(-\frac{(\varrho + v)^2}{2(q + u)} - \frac{v^2}{2u}\right) dv &= \frac{\sqrt{\pi(q + u)u}}{\sqrt{2(q + 2u)}} \exp\left(-\frac{\varrho^2}{2(q + 2u)}\right) \\ &\times \left( \tilde{\Phi}\left(\frac{(q + 2u)b + \varrho u}{\sqrt{2(q + 2u)(q + u)u}}\right) - \tilde{\Phi}\left(\frac{(q + 2u)a + \varrho u}{\sqrt{2(q + 2u)(q + u)u}}\right) \right), \end{aligned}$$

where  $\tilde{\Phi}(x) := 2\Phi(\sqrt{2}x) - 1$ ,  $x \in \mathbb{R}$ , is the Gauss error function defined with the help of the cdf  $\Phi(x)$  of the standard normal distribution. Hence

$$\begin{aligned} H_{\alpha, \beta}(k_1, \ell_1, k_2, \ell_2) &\leq \frac{2}{\sqrt{2\pi}(\alpha + \beta)(1 - \alpha)(1 - \beta)} \\ &\times \int_{b_{1,1}}^{b_{k_1 \wedge k_2, \ell_1 \wedge \ell_2}} \frac{1}{\sqrt{b_{|k_1 - k_2|, |\ell_1 - \ell_2|} + 2u}} \exp\left(-\frac{a_{|k_1 - k_2|, |\ell_1 - \ell_2|}^2}{2(b_{|k_1 - k_2|, |\ell_1 - \ell_2|} + 2u)}\right) du \\ &\leq \frac{2}{\sqrt{2\pi}(\alpha + \beta)(1 - \alpha)(1 - \beta)} \int_{b_{1,1}}^{b_{k_1 \wedge k_2, \ell_1 \wedge \ell_2}} \frac{1}{\sqrt{b_{|k_1 - k_2|, |\ell_1 - \ell_2|} + 2u}} du \\ &\leq \frac{\sqrt{2\alpha(1 - \alpha)(k_1 + k_2) + 2\beta(1 - \beta)(\ell_1 + \ell_2)}}{\sqrt{\pi}(\alpha + \beta)(1 - \alpha)(1 - \beta)} \leq \frac{\sqrt{k_1 + \ell_1 + k_2 + \ell_2}}{\sqrt{2\pi}(\alpha + \beta)(1 - \alpha)(1 - \beta)} \end{aligned}$$

that together with (2.4) implies the second statement of the theorem. Cases  $k_1 \leq k_2$ ,  $\ell_1 > \ell_2$  and  $k_1 > k_2$ ,  $\ell_1 \leq \ell_2$  can be handled in a similar way.

Further, let  $\alpha\beta\gamma < 0$  and  $|\alpha| = 1, |\beta| = |\gamma| < 1$  or  $|\beta| = 1, |\alpha| = |\gamma| < 1$ . In this case  $-\gamma/(\alpha\beta) = 1$ . As for  $n, m \in \mathbb{N}$  one has  $F(-n, -m; -n - m; 1) = \binom{m+n}{n}^{-1}$ , representation (2.2) implies

$$\text{Cov}(X_{k_1, \ell_1}, X_{k_2, \ell_2}) = \sum_{i=1}^{k_1 \wedge k_2} \sum_{j=1}^{\ell_1 \wedge \ell_2} \alpha^{k_1+k_2-2i} \beta^{\ell_1+\ell_2-2j}. \quad (2.5)$$

Obviously, (2.5) also holds if  $|\alpha| = 1, \beta = \gamma = 0$  or  $|\beta| = 1, \alpha = \gamma = 0$ . Hence, e.g. if  $|\alpha| = 1, |\beta| = |\gamma| < 1$

$$|\text{Cov}(X_{k_1, \ell_1}, X_{k_2, \ell_2})| \leq (k_1 \wedge k_2) \sum_{j=1}^{\ell_1 \wedge \ell_2} |\gamma|^{\ell_1+\ell_2-2j} \leq (k_1 \wedge k_2) |\gamma|^{|\ell_1-\ell_2|} \sum_{j=0}^{\ell_1 \wedge \ell_2 - 1} \gamma^{2j} \leq (k_1 \wedge k_2) \frac{|\gamma|^{|\ell_1-\ell_2|}}{1-\gamma^2}.$$

Finally, if  $\alpha\beta\gamma = -1$  and  $|\alpha| = |\beta| = |\gamma| = 1$  then  $-\gamma/(\alpha\beta) = 1$ , so

$$\text{Cov}(X_{k_1, \ell_1}, X_{k_2, \ell_2}) = \sum_{i=1}^{k_1 \wedge k_2} \sum_{j=1}^{\ell_1 \wedge \ell_2} \alpha^{k_1+k_2} \beta^{\ell_1+\ell_2} = (k_1 \wedge k_2)(\ell_1 \wedge \ell_2) \alpha^{k_1-k_2} \beta^{|\ell_1-\ell_2|} \quad (2.6)$$

that completes the proof.  $\square$

### 3 Proof of Theorem 1.1

According to the results of the Introduction we may assume  $\alpha \geq 0, \beta \geq 0$  and  $\gamma \geq 0$  if  $\alpha\beta\gamma \geq 0$  and  $\alpha > 0, \beta > 0$  and  $\gamma < 0$ , otherwise.

Let  $0 \leq \alpha, \beta < 1$  and  $|\gamma| < 1, |1 + \alpha^2 - \beta^2 - \gamma^2| > 2|\alpha + \beta\gamma|$  and  $1 - \beta^2 > |\alpha + \beta\gamma|$ . Representation (2.1) directly implies

$$\lim_{n \rightarrow \infty} \text{Var}(Y^{(n)}(s, t)) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( G(i, j; \alpha, \beta, \gamma) \right)^2. \quad (3.1)$$

To show that the right hand side of (3.1) equals  $\sigma_{\alpha, \beta, \gamma}^2$  consider the stationary solution  $X_{k, \ell}^*$  of the equation

$$X_{k, \ell}^* = \alpha X_{k-1, \ell}^* + \beta X_{k, \ell-1}^* + \gamma X_{k-1, \ell-1}^* + \varepsilon_{k, \ell}^*, \quad k, \ell \in \mathbb{Z},$$

where  $\{\varepsilon_{k, \ell}^* : k, \ell \in \mathbb{Z}\}$  are independent random variables with  $\mathbb{E} \varepsilon_{k, \ell}^* = 0$  and  $\text{Var} \varepsilon_{k, \ell}^* = 1$ . As the model is stable,  $X_{k, \ell}^*$  has the following  $\mathbf{L}^2$ -convergent infinite moving average representation (see Tjøstheim (1978, Lemma 5.1))

$$X_{k, \ell}^* = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} G(i, j; \alpha, \beta, \gamma) \varepsilon_{k-i, \ell-j}.$$

Hence,

$$\text{Var}(X_{k,\ell}^*) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( G(i, j; \alpha, \beta, \gamma) \right)^2.$$

On the other hand, using the results of Basu and Reinsel (1993) one can easily show that  $\text{Var}(X_{k,\ell}) = \sigma_{\alpha,\beta,\gamma}^2$ .

Further, let  $0 \leq \alpha, \beta < 1$ ,  $|\gamma| < 1$ , and  $\alpha + \beta + \gamma = 1$ . Corollary 2.2 and (2.1) imply

$$\text{Var}(Y^{(n)}(s, t)) = \sum_{k=0}^{[ns]-1} \sum_{\ell=0}^{[nt]-1} \mathbf{P}^2 \left( \xi_k^{(\alpha)} + \eta_{\ell}^{(1-\beta)} = k \right).$$

Hence, to find the limit on  $n^{-1/2} \text{Var}(Y^{(n)}(s, t))$  as  $n \rightarrow \infty$ , one can use the local version of the central limit theorem given in Lemma 2.3 that yields approximation

$$\begin{aligned} \text{Var}(Y^{(n)}(s, t)) &\approx \tilde{E}_{\alpha,\beta}^{(n)}(s, t) := \sum_{k=1}^{[ns]-1} \sum_{\ell=1}^{[nt]-1} \frac{1}{2\pi b_{k,\ell}} \exp(-x_{k,k,\ell}^2) \\ &= \int_1^{[ns]} \int_1^{[nt]} \frac{1}{2\pi b_{[y],[z]}} \exp(-x_{[y],[y],[z]}^2) dz dy. \end{aligned}$$

Direct calculations show that for the error

$$\tilde{\Delta}_{\alpha,\beta}^{(n)}(s, t) := \text{Var}(Y^{(n)}(s, t)) - \tilde{E}_{\alpha,\beta}^{(n)}(s, t)$$

of the approximation we have

$$|\tilde{\Delta}_{\alpha,\beta}^{(n)}(s, t)| \leq C(\alpha, \beta) \left( 1 + \sum_{k=2}^{[ns]-1} \sum_{\ell=2}^{[nt]-1} \frac{1}{b_{k,\ell}^2} + \sum_{k=2}^{[ns]-1} \sum_{\ell=2}^{[nt]-1} \frac{1}{\sqrt{2\pi} b_{k,\ell}^{3/2}} \exp\left(-\frac{x_{k,k,\ell}^2}{2}\right) \right), \quad (3.2)$$

where  $C(\alpha, \beta)$  is a positive constant. Now, similarly to the proof of Theorem 2.4 one can verify that

$$\sum_{k=2}^{[ns]-1} \sum_{\ell=2}^{[nt]-1} \frac{1}{b_{k,\ell}^2} \leq \frac{\ln(\alpha(1-\alpha)([ns]-1) + \beta(1-\beta))}{\alpha\beta(1-\alpha)(1-\beta)} \leq \frac{\ln([ns])}{\alpha\beta(1-\alpha)(1-\beta)}. \quad (3.3)$$

Further,

$$\begin{aligned} \sum_{k=2}^{[ns]-1} \sum_{\ell=2}^{[nt]-1} \frac{1}{b_{k,\ell}^{3/2}} \exp\left(-\frac{x_{k,k,\ell}^2}{2}\right) &\leq 4 \int_1^{[ns]} \int_1^{[nt]} \frac{1}{b_{y,z}^{3/2}} \exp\left(-\frac{x_{y,y,z}^2}{2}\right) dz dy \\ &\leq \frac{4}{(\alpha + \beta)(1-\alpha)(1-\beta)} \int_{b_{1,1}}^{b_{[ns],[nt]}} \int_{a_{1,[nt]}}^{a_{[ns],1}} \frac{1}{u^{3/2}} \exp\left(-\frac{v^2}{2u}\right) dv du. \end{aligned} \quad (3.4)$$

Again, for some real constants  $a < b$  and  $m > 0$

$$\int_a^b \exp\left(-\frac{v^2}{mu}\right) dv = \frac{\sqrt{\pi mu}}{2} \left( \tilde{\Phi}\left(\frac{b}{\sqrt{mu}}\right) - \tilde{\Phi}\left(\frac{a}{\sqrt{mu}}\right) \right) \quad (3.5)$$

holds, so using (3.4) and (3.5) with  $m = 2$  we have

$$\sum_{k=2}^{[ns]-1} \sum_{\ell=2}^{[nt]-1} \frac{1}{b_{k,\ell}^{3/2}} \exp\left(-\frac{x_{k,k,\ell}^2}{2}\right) \leq \frac{4\sqrt{2\pi}}{(\alpha+\beta)(1-\alpha)(1-\beta)} \ln\left(\frac{[ns] + [nt]}{b_{1,1}}\right)$$

that together with (3.2) and (3.3) implies

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} \tilde{\Delta}_{\alpha,\beta}^{(n)}(s, t) = 0.$$

Hence,  $n^{-1/2} \text{Var}(Y^{(n)}(s, t))$  and  $n^{-1/2} \tilde{E}_{\alpha,\beta}^{(n)}(s, t)$  have the same limit as  $n \rightarrow \infty$ .

Now, let

$$\Delta_{\alpha,\beta}^{(n)}(s, t) := \tilde{E}_{\alpha,\beta}^{(n)}(s, t) - E_{\alpha,\beta}^{(n)}(s, t),$$

where

$$E_{\alpha,\beta}^{(n)}(s, t) := \int_1^{[ns]} \int_1^{[nt]} \frac{1}{2\pi b_{y,z}} \exp(-x_{y,y,z}^2) dz dy.$$

Obviously,

$$\Delta_{\alpha,\beta}^{(n)}(s, t) = \Delta_{\alpha,\beta}^{(n,1)}(s, t) + \Delta_{\alpha,\beta}^{(n,2)}(s, t), \quad (3.6)$$

where

$$\begin{aligned} \Delta_{\alpha,\beta}^{(n,1)}(s, t) &:= \frac{1}{2\pi} \int_1^{[ns]} \int_1^{[nt]} \left( \frac{1}{b_{[y],[z]}} \exp\left(-\frac{a_{[y],[z]}^2}{b_{[y],[z]}}\right) - \frac{1}{b_{y,z}} \exp\left(-\frac{a_{[y],[z]}^2}{b_{y,z}}\right) \right) dz dy, \\ \Delta_{\alpha,\beta}^{(n,2)}(s, t) &:= \frac{1}{2\pi} \int_1^{[ns]} \int_1^{[nt]} \left( \frac{1}{b_{y,z}} \exp\left(-\frac{a_{[y],[z]}^2}{b_{y,z}}\right) - \frac{1}{b_{y,z}} \exp\left(-\frac{a_{y,z}^2}{b_{y,z}}\right) \right) dz dy. \end{aligned}$$

As  $|z - [z]| < 1$ ,  $z \in \mathbb{R}$ , and for  $z \geq 0$  we have  $z \exp(-z) \leq 1$ , and  $|1 - \exp(-z)| \leq |z|$ , while for  $z \geq 1$ ,  $[z] > z/2$  holds, after short straightforward calculations (see also (3.3)) we obtain

$$|\Delta_{\alpha,\beta}^{(n,1)}(s, t)| \leq \int_1^{[ns]} \int_1^{[nt]} \frac{b_{1,1}}{\pi b_{y,z}^2} dz dy \leq \frac{\ln([ns] + 1)}{2\pi\alpha\beta(1-\alpha)(1-\beta)}. \quad (3.7)$$

Further, using similar ideas as in the proof of (3.7) we have

$$\begin{aligned}
|\Delta_{\alpha,\beta}^{(n,2)}(s,t)| &\leq \frac{1}{2\pi} \int_1^{[ns]} \int_1^{[nt]} \frac{|a_{y,z}^2 - a_{[y],[z]}^2|}{b_{y,z}^2} \exp\left(-\frac{a_{y,z}^2 \wedge a_{[y],[z]}^2}{b_{y,z}}\right) dz dy \leq \frac{1}{2\pi} \int_1^{[ns]} \int_1^{[nt]} \frac{(2-\alpha-\beta)^2}{b_{y,z}^2} dz dy \\
&\quad + \frac{2-\alpha-\beta}{\pi} \int_1^{[ns]} \int_1^{[nt]} \frac{|a_{y,z}| \wedge |a_{[y],[z]}|}{b_{y,z}^2} \exp\left(-\frac{a_{y,z}^2 \wedge a_{[y],[z]}^2}{b_{y,z}}\right) dz dy \\
&\leq \frac{4}{\pi} \int_1^{[ns]} \int_1^{[nt]} \frac{1}{b_{y,z}^2} dz dy + \frac{2}{\pi} \int_1^{[ns]} \int_1^{[nt]} \mathcal{X}_{\{|a_{y,z}| \wedge |a_{[y],[z]}| \geq 1\}} \frac{1}{b_{y,z}(|a_{y,z}| \wedge |a_{[y],[z]}|)} dz dy \\
&\leq \frac{8 \ln([ns] + 1)}{\pi \alpha \beta (1-\alpha)(1-\beta)} + \frac{2}{\pi} \int_1^{[ns]} \int_1^{[nt]} \mathcal{X}_{\{|a_{y,z}| \geq 1\}} \frac{1}{b_{y,z}|a_{y,z}|} dz dy \\
&\leq \frac{8 \ln([ns] + 1)}{\pi \alpha \beta (1-\alpha)(1-\beta)} + \frac{2}{\pi(\alpha + \beta)(1-\alpha)(1-\beta)} \int_{b_{1,1}}^{b_{[ns],[nt]}} \frac{1}{u} du \int_{a_{1,[nt]}}^{a_{[ns],1}} \mathcal{X}_{\{|v| \geq 1\}} \frac{1}{|v|} dv \\
&\leq \frac{8 \ln([ns] + 1)}{\pi \alpha \beta (1-\alpha)(1-\beta)} + \frac{4 \ln([ns] + [nt])}{\pi(\alpha + \beta)(1-\alpha)(1-\beta)} \ln\left(\frac{[ns] + [nt]}{b_{1,1}}\right),
\end{aligned}$$

where  $\mathcal{X}_H$  denotes the indicator function of a set  $H$ , that together with (3.6) and (3.7) implies

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} \Delta_{\alpha,\beta}^{(n)}(s,t) = 0.$$

Hence,  $n^{-1/2} \text{Var}(Y^{(n)}(s,t))$  and  $n^{-1/2} E_{\alpha,\beta}^{(n)}(s,t)$  have the same limit as  $n \rightarrow \infty$ .

Now, consider first the case  $\alpha(1-\alpha)s \leq \beta(1-\beta)t$  implying  $\alpha(1-\alpha)[ns] + \beta(1-\beta) \leq \alpha(1-\alpha) + \beta(1-\beta)[nt]$ , if  $n$  is large enough. In this case

$$E_{\alpha,\beta}^{(n)}(s,t) = \frac{1}{(\alpha + \beta)(1-\alpha)(1-\beta)} (E_{\alpha,\beta}^{(n,1)}(s,t) + E_{\alpha,\beta}^{(n,2)}(s,t) + E_{\alpha,\beta}^{(n,3)}(s,t)),$$

where

$$\begin{aligned}
E_{\alpha,\beta}^{(n,1)}(s,t) &:= \frac{1}{2\pi} \int_{b_{1,1}}^{b_{[ns],1}} \int_{-u/\beta + (\alpha+\beta)(1-\alpha)/\beta}^{u/\alpha - (\alpha+\beta)(1-\beta)/\alpha} \frac{1}{u} \exp\left(-\frac{v^2}{u}\right) dv du, \\
E_{\alpha,\beta}^{(n,2)}(s,t) &:= \frac{1}{2\pi} \int_{b_{[ns],1}}^{b_{1,[nt]}} \int_{-u/\beta + (\alpha+\beta)(1-\alpha)[ns]/\beta}^{-u/\beta + (\alpha+\beta)(1-\alpha)/\beta} \frac{1}{u} \exp\left(-\frac{v^2}{u}\right) dv du, \\
E_{\alpha,\beta}^{(n,3)}(s,t) &:= \frac{1}{2\pi} \int_{b_{1,[nt]}}^{b_{[ns],[nt]}} \int_{u/\alpha - (\alpha+\beta)(1-\beta)[nt]/\alpha}^{-u/\beta + (\alpha+\beta)(1-\alpha)[ns]/\beta} \frac{1}{u} \exp\left(-\frac{v^2}{u}\right) dv du.
\end{aligned}$$

Using (3.5) with  $m = 1$ , as  $\tilde{\Phi}(-x) = -\tilde{\Phi}(x)$ , we have

$$\begin{aligned}
E_{\alpha,\beta}^{(n,1)}(s, t) &= \frac{1}{2\sqrt{\pi}} \int_{b_{1,1}}^{b_{[ns],1}} \frac{1}{2\sqrt{u}} \left( \tilde{\Phi}\left(\frac{\sqrt{u}}{\alpha} - \frac{(\alpha+\beta)(1-\beta)}{\sqrt{u}\alpha}\right) + \tilde{\Phi}\left(\frac{\sqrt{u}}{\beta} - \frac{(\alpha+\beta)(1-\alpha)}{\sqrt{u}\beta}\right) \right) du \\
&= \frac{1}{2\sqrt{\pi}} \int_{\sqrt{b_{1,1}}}^{\sqrt{b_{[ns],1}}} \left( \tilde{\Phi}\left(\frac{w}{\alpha} - \frac{(\alpha+\beta)(1-\beta)}{w\alpha}\right) + \tilde{\Phi}\left(\frac{w}{\beta} - \frac{(\alpha+\beta)(1-\alpha)}{w\beta}\right) \right) dw, \\
E_{\alpha,\beta}^{(n,2)}(s, t) &= \frac{1}{2\sqrt{\pi}} \int_{\sqrt{b_{[ns],1}}}^{\sqrt{b_{1,[nt]}}} \left( \tilde{\Phi}\left(-\frac{w}{\beta} + \frac{(\alpha+\beta)(1-\alpha)[ns]}{w\beta}\right) + \tilde{\Phi}\left(\frac{w}{\beta} - \frac{(\alpha+\beta)(1-\alpha)}{w\beta}\right) \right) dw, \\
E_{\alpha,\beta}^{(n,3)}(s, t) &= \frac{1}{2\sqrt{\pi}} \int_{\sqrt{b_{1,[nt]}}}^{\sqrt{b_{[ns],[nt]}}} \left( \tilde{\Phi}\left(-\frac{w}{\beta} + \frac{(\alpha+\beta)(1-\alpha)[ns]}{w\beta}\right) - \tilde{\Phi}\left(\frac{w}{\alpha} - \frac{(\alpha+\beta)(1-\beta)[nt]}{w\alpha}\right) \right) dw.
\end{aligned}$$

Combining similar terms we obtain

$$E_{\alpha,\beta}^{(n)}(s, t) = \frac{1}{2\sqrt{\pi}(\alpha+\beta)(1-\alpha)(1-\beta)} (F_{\alpha,\beta}^{(n,1)}(s, t) + F_{\alpha,\beta}^{(n,2)}(s, t) + F_{\alpha,\beta}^{(n,3)}(s, t) + F_{\alpha,\beta}^{(n,4)}(s, t)), \quad (3.8)$$

where

$$\begin{aligned}
F_{\alpha,\beta}^{(n,1)}(s, t) &:= \int_{\sqrt{b_{1,1}}}^{\sqrt{b_{[ns],1}}} \tilde{\Phi}\left(\frac{w}{\alpha} - \frac{(\alpha+\beta)(1-\beta)}{w\alpha}\right) dw, & F_{\alpha,\beta}^{(n,3)}(s, t) &:= \int_{\sqrt{b_{[ns],1}}}^{\sqrt{b_{[ns],[nt]}}} \tilde{\Phi}\left(\frac{(\alpha+\beta)(1-\alpha)[ns]}{w\beta} - \frac{w}{\beta}\right) dw, \\
F_{\alpha,\beta}^{(n,2)}(s, t) &:= \int_{\sqrt{b_{1,1}}}^{\sqrt{b_{1,[nt]}}} \tilde{\Phi}\left(\frac{w}{\beta} - \frac{(\alpha+\beta)(1-\alpha)}{w\beta}\right) dw, & F_{\alpha,\beta}^{(n,4)}(s, t) &:= \int_{\sqrt{b_{1,[nt]}}}^{\sqrt{b_{[ns],[nt]}}} \tilde{\Phi}\left(\frac{(\alpha+\beta)(1-\beta)[nt]}{w\alpha} - \frac{w}{\alpha}\right) dw.
\end{aligned}$$

Let

$$G_{\alpha,\beta}^{(n,1)}(s, t) := \int_{\sqrt{b_{1,1}}}^{\sqrt{b_{[ns],1}}} \tilde{\Phi}\left(\frac{w}{\alpha}\right) dw, \quad G_{\alpha,\beta}^{(n,2)}(s, t) := \int_{\sqrt{b_{1,1}}}^{\sqrt{b_{1,[nt]}}} \tilde{\Phi}\left(\frac{w}{\beta}\right) dw.$$

Short calculation shows that

$$\frac{1}{n^{1/2}} \left| F_{\alpha,\beta}^{(n,1)}(s, t) - G_{\alpha,\beta}^{(n,1)}(s, t) \right| \leq \frac{2(\alpha+\beta)(1-\beta)}{\alpha\sqrt{\pi n}} \int_{\sqrt{b_{1,1}}}^{\sqrt{b_{[ns],1}}} \frac{1}{w} dw \leq \frac{1}{\alpha\sqrt{\pi n}} \ln \left( \frac{[ns]+1}{b_{1,1}} \right) \rightarrow 0 \quad (3.9)$$

as  $n \rightarrow \infty$ . Further, for  $a < b$  we have

$$\int_a^b \tilde{\Phi}\left(\frac{w}{\alpha}\right) dw = \frac{\alpha}{\sqrt{\pi}} \left( \exp\left(-\frac{b^2}{\alpha^2}\right) - \exp\left(-\frac{a^2}{\alpha^2}\right) \right) + b\tilde{\Phi}\left(\frac{b}{\alpha}\right) - a\tilde{\Phi}\left(\frac{a}{\alpha}\right),$$

so

$$G_{\alpha,\beta}^{(n,1)}(s,t) = b_{[ns],1}^{1/2} \tilde{\Phi}\left(\frac{b_{[ns],1}^{1/2}}{\alpha}\right) - b_{1,1}^{1/2} \tilde{\Phi}\left(\frac{b_{1,1}^{1/2}}{\alpha}\right) + \frac{\alpha}{\sqrt{\pi}} \left( \exp\left(-\frac{b_{[ns],1}}{\alpha^2}\right) - \exp\left(-\frac{b_{1,1}}{\alpha^2}\right) \right)$$

that together with (3.9) implies

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} F_{\alpha,\beta}^{(n,1)}(s,t) = \lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} G_{\alpha,\beta}^{(n,1)}(s,t) = (\alpha(1-\alpha)s)^{1/2}. \quad (3.10)$$

Similarly,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} F_{\alpha,\beta}^{(n,2)}(s,t) = \lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} G_{\alpha,\beta}^{(n,2)}(s,t) = (\beta(1-\beta)t)^{1/2}. \quad (3.11)$$

To determine the limit of  $n^{-1/2} F_{\alpha,\beta}^{(n,3)}(s,t)$  assume first that  $(1-\beta)t < (1-\alpha)s$  implying  $(1-\beta)[nt] < (1-\alpha)[ns]$  if  $n$  is large enough. On the one hand we have

$$\frac{1}{n^{1/2}} F_{\alpha,\beta}^{(n,3)}(s,t) \geq \tilde{\Phi}\left(\frac{(1-\alpha)[ns] - (1-\beta)[nt]}{\sqrt{b_{[ns],[nt]}}}\right) \frac{\sqrt{b_{[ns],[nt]}} - \sqrt{b_{[ns],1}}}{\sqrt{n}} \rightarrow \sqrt{b_{s,t}} - \sqrt{b_{s,0}}$$

as  $n \rightarrow \infty$ . On the other hand

$$\frac{1}{n^{1/2}} F_{\alpha,\beta}^{(n,3)}(s,t) \leq \frac{\sqrt{b_{[ns],[nt]}} - \sqrt{b_{[ns],1}}}{\sqrt{n}} \rightarrow \sqrt{b_{s,t}} - \sqrt{b_{s,0}}$$

as  $n \rightarrow \infty$ , so

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} F_{\alpha,\beta}^{(n,3)}(s,t) = \sqrt{b_{s,t}} - \sqrt{b_{s,0}} = (\alpha(1-\alpha)s + \beta(1-\beta)t)^{1/2} - (\alpha(1-\alpha)s)^{1/2}. \quad (3.12)$$

If  $(1-\beta)t \geq (1-\alpha)s$  we split the domain of integration in  $F_{\alpha,\beta}^{(n,3)}(s,t)$  into two parts, that is  $F_{\alpha,\beta}^{(n,3)}(s,t) = F_{\alpha,\beta}^{(n,3,1)}(s,t) + F_{\alpha,\beta}^{(n,3,2)}(s,t)$  where

$$F_{\alpha,\beta}^{(n,3,1)}(s,t) := \int_{\sqrt{b_{[ns],1}}}^{\sqrt{(\alpha+\beta)(1-\beta)[ns]}} \tilde{\Phi}\left(\frac{(\alpha+\beta)(1-\alpha)[ns]}{w\beta} - \frac{w}{\beta}\right) dw,$$

$$F_{\alpha,\beta}^{(n,3,2)}(s,t) := \int_{\sqrt{(\alpha+\beta)(1-\beta)[ns]}}^{\sqrt{b_{[ns],[nt]}}} \tilde{\Phi}\left(\frac{(\alpha+\beta)(1-\alpha)[ns]}{w\beta} - \frac{w}{\beta}\right) dw.$$



Again, on the one hand we have

$$\begin{aligned}
\frac{1}{n^{1/2}} F_{\alpha,\beta}^{(n,3,1)}(s, t) &\geq \frac{1}{n^{1/2}} \int_{\sqrt{b_{[ns],1}}}^{\sqrt{(\alpha+\beta)(1-\beta)[ns]}} \tilde{\Phi}\left(\frac{\sqrt{(\alpha+\beta)(1-\alpha)[ns]} - w}{\beta}\right) dw \\
&= \frac{\sqrt{(\alpha+\beta)(1-\beta)[ns]} - \sqrt{b_{[ns],1}}}{\sqrt{n}} \tilde{\Phi}\left(\frac{\sqrt{(\alpha+\beta)(1-\alpha)[ns]} - \sqrt{b_{[ns],1}}}{\beta}\right) \\
&\quad + \frac{\beta}{\sqrt{n\pi}} \left( \exp\left(-\frac{(\sqrt{(\alpha+\beta)(1-\beta)[ns]} - \sqrt{b_{[ns],1}})^2}{\beta^2}\right) - 1 \right) \rightarrow \sqrt{(\alpha+\beta)(1-\beta)s} - \sqrt{b_{s,0}}
\end{aligned} \tag{3.13}$$

as  $n \rightarrow \infty$ . On the other hand

$$\frac{1}{n^{1/2}} F_{\alpha,\beta}^{(n,3,1)}(s, t) \leq \frac{\sqrt{(\alpha+\beta)(1-\beta)[ns]} - \sqrt{b_{[ns],1}}}{\sqrt{n}} \rightarrow \sqrt{(\alpha+\beta)(1-\beta)s} - \sqrt{b_{s,0}}$$

as  $n \rightarrow \infty$ , so

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} F_{\alpha,\beta}^{(n,3,1)}(s, t) = ((\alpha + \beta)(1 - \alpha)s)^{1/2} - (\alpha(1 - \alpha)s)^{1/2}. \tag{3.14}$$

Similarly to (3.13) one can also show

$$\begin{aligned}
-\frac{1}{n^{1/2}} F_{\alpha,\beta}^{(n,3,2)}(s, t) &\geq \frac{1}{n^{1/2}} \int_{\sqrt{(\alpha+\beta)(1-\beta)[ns]}}^{\sqrt{b_{[ns],[nt]}}} \tilde{\Phi}\left(\frac{w - \sqrt{(\alpha+\beta)(1-\alpha)[ns]}}{\beta}\right) dw \\
&\rightarrow \sqrt{b_{s,t}} - \sqrt{(\alpha+\beta)(1-\beta)s},
\end{aligned}$$

and we also have

$$-\frac{1}{n^{1/2}} F_{\alpha,\beta}^{(n,3,2)}(s, t) \leq \frac{\sqrt{b_{[ns],[nt]}} - \sqrt{(\alpha+\beta)(1-\beta)[ns]}}{\sqrt{n}} \rightarrow \sqrt{b_{s,t}} - \sqrt{(\alpha+\beta)(1-\beta)s}$$

as  $n \rightarrow \infty$  implying

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} F_{\alpha,\beta}^{(n,3,2)}(s, t) = ((\alpha + \beta)(1 - \alpha)s)^{1/2} - (\alpha(1 - \alpha)s + \beta(1 - \beta)t)^{1/2}. \tag{3.15}$$

Thus, by summing the limits in (3.14) and (3.15) we obtain that for  $(1 - \beta)t \geq (1 - \alpha)s$

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} F_{\alpha,\beta}^{(n,3)}(s, t) = 2((\alpha+\beta)(1-\alpha)s)^{1/2} - (\alpha(1-\alpha)s + \beta(1-\beta)t)^{1/2} - (\alpha(1-\alpha)s)^{1/2}. \tag{3.16}$$

Finally, the asymptotic behaviour of  $n^{-1/2} F_{\alpha,\beta}^{(n,4)}(s, t)$  in some sense a complementary of the behaviour of  $n^{-1/2} F_{\alpha,\beta}^{(n,3)}(s, t)$ . In the same way as (3.12) is proved one can show that if  $(1 - \beta)t > (1 - \alpha)s$

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} F_{\alpha,\beta}^{(n,4)}(s, t) = (\alpha(1 - \alpha)s + \beta(1 - \beta)t)^{1/2} - (\beta(1 - \beta)t)^{1/2}. \tag{3.17}$$

In case  $(1 - \beta)t \leq (1 - \alpha)s$  the domain of integration in  $F_{\alpha,\beta}^{(n,4)}(s, t)$  has to be split at  $\sqrt{(\alpha + \beta)(1 - \beta)[nt]}$  to obtain

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} F_{\alpha,\beta}^{(n,4)}(s, t) = 2((\alpha + \beta)(1 - \beta)t)^{1/2} - (\alpha(1 - \alpha)s + \beta(1 - \beta)t)^{1/2} - (\beta(1 - \beta)t)^{1/2}. \quad (3.18)$$

Hence, in case  $\alpha(1 - \alpha)s \leq \beta(1 - \beta)t$  equation (3.8) and limits (3.10) – (3.12) and (3.16) – (3.18) imply

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} \text{Var}(Y^{(n)}(s, t)) = \lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} E_{\alpha,\beta}^{(n)}(s, t) = \frac{((1 - \alpha)s)^{1/2} \wedge ((1 - \beta)t)^{1/2}}{\pi^{1/2}(\alpha + \beta)^{1/2}(1 - \alpha)(1 - \beta)}. \quad (3.19)$$

If  $\alpha(1 - \alpha)s > \beta(1 - \beta)t$  we have

$$E_{\alpha,\beta}^{(n)}(s, t) = \frac{1}{(\alpha + \beta)(1 - \alpha)(1 - \beta)} (E_{\alpha,\beta}^{(n,1)}(s, t) + E_{\alpha,\beta}^{(n,2)}(s, t) + E_{\alpha,\beta}^{(n,3)}(s, t)),$$

with

$$\begin{aligned} E_{\alpha,\beta}^{(n,1)}(s, t) &:= \frac{1}{2\pi} \int_{b_{1,1}}^{b_{1,[nt]}} \int_{-u/\beta + (\alpha + \beta)(1 - \alpha)/\beta}^{u/\alpha - (\alpha + \beta)(1 - \beta)/\alpha} \frac{1}{u} \exp\left(-\frac{v^2}{u}\right) dv du, \\ E_{\alpha,\beta}^{(n,2)}(s, t) &:= \frac{1}{2\pi} \int_{b_{1,[nt]}}^{b_{[ns],1}} \int_{u/\alpha - (\alpha + \beta)(1 - \beta)[nt]/\alpha}^{u/\alpha - (\alpha + \beta)(1 - \beta)/\alpha} \frac{1}{u} \exp\left(-\frac{v^2}{u}\right) dv du, \\ E_{\alpha,\beta}^{(n,3)}(s, t) &:= \frac{1}{2\pi} \int_{b_{[ns],1}}^{b_{[ns],[nt]}} \int_{u/\alpha - (\alpha + \beta)(1 - \beta)[nt]/\alpha}^{-u/\beta + (\alpha + \beta)(1 - \alpha)[ns]/\beta} \frac{1}{u} \exp\left(-\frac{v^2}{u}\right) dv du \end{aligned}$$

and (3.19) can be proved similarly to the other case.

Now, if  $\alpha\beta\gamma \leq 0$  and  $|\alpha| = 1$ ,  $|\beta| = |\gamma| < 1$  or  $|\beta| = 1$ ,  $|\alpha| = |\gamma| < 1$  using (2.5) we have

$$\frac{1}{n} \text{Var}(Y^{(n)}(s, t)) = \frac{[ns]}{n} \frac{1 - \gamma^{2[nt]}}{1 - \gamma^2} \rightarrow \frac{s}{1 - \gamma^2} \quad \text{or} \quad \frac{1}{n} \text{Var}(Y^{(n)}(s, t)) = \frac{[nt]}{n} \frac{1 - \gamma^{2[ns]}}{1 - \gamma^2} \rightarrow \frac{t}{1 - \gamma^2},$$

respectively, as  $n \rightarrow \infty$ .

At the end, if  $\alpha = \beta = -\gamma = 1$  the statement directly follows from Theorem 2.4.  $\square$

## 4 Remarks on missing cases

The results of Theorem 1.1 do not cover the cases when  $|\alpha| < 1$ ,  $|\beta| < 1$ ,  $|\gamma| \leq 1$ , either  $\alpha\beta\gamma < 0$  or  $\alpha = \beta = 0$  is satisfied, and  $|\alpha| - |\beta| + |\gamma| = 1$  or  $-|\alpha| + |\beta| + |\gamma| = 1$  holds.

For  $|\gamma| < 1$  the above conditions yield two subcases of Case A, while for  $|\gamma| = 1$  we have a subcase of Case B.

In the trivial case  $\alpha = \beta = 0$  and  $|\gamma| = 1$  using directly (1.2) it is easy to see that  $\text{Var}(X_{k,\ell}) = k \wedge \ell$ , hence

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Var}(Y^{(n)}(s, t)) = s \wedge t. \quad (4.1)$$

If  $\alpha\beta\gamma < 0$  according to the results of the Introduction it suffices to consider  $0 < \alpha, \beta < 1$  and  $-1 \leq \gamma < 0$  and assume  $\alpha - \beta - \gamma = 1$  or  $-\alpha + \beta - \gamma = 1$ . As the first row of (2.3) holds for all positive  $\alpha$  and  $\beta$ ,

$$G(m, n; \alpha, \beta, \gamma) = \sum_{r=0}^{m \wedge n} \binom{n}{r} \binom{m}{r} \alpha^{m-r} \beta^{n-r} (\alpha\beta + \gamma)^r,$$

where

$$\alpha\beta + \gamma = \begin{cases} (1 + \beta)(\alpha - 1) < 0, & \text{if } \alpha - \beta - \gamma = 1, \\ (1 + \alpha)(\beta - 1) < 0, & \text{if } -\alpha + \beta - \gamma = 1. \end{cases}$$

Hence, using notations of Lemma 2.1, for  $\alpha - \beta - \gamma = 1$  we have

$$G(m, n; \alpha, \beta, \gamma) = (1 + 2\beta)^n \sum_{r=0}^{m \wedge n} (-1)^r \mathbb{P}(\xi_m^{(\alpha)} = m - r) \mathbb{P}\left(\eta_n^{\left(\frac{1+\beta}{1+2\beta}\right)} = r\right), \quad (4.2)$$

while for  $-\alpha + \beta - \gamma = 1$

$$G(m, n; \alpha, \beta, \gamma) = (1 + 2\alpha)^m \sum_{r=0}^{m \wedge n} (-1)^r \mathbb{P}(\xi_n^{(\beta)} = n - r) \mathbb{P}\left(\eta_m^{\left(\frac{1+\alpha}{1+2\alpha}\right)} = r\right) \quad (4.3)$$

holds. This means that results similar to Corollary 2.2 can not be obtained. Moreover, the exponential terms before the sums in (4.2) and (4.3) do not allow us to use Lemma 2.3 for separate approximations of the probabilities behind the sums.

Finally, in case  $\alpha = \beta < 1$ ,  $\gamma = -1$  short calculation shows (Szegő, 1939)

$$G(m, n; \alpha, \alpha, -1) = \alpha^{|m-n|} P_{m \wedge n}^{(0, |m-n|)} (2\alpha^2 - 1),$$

so using notation  $\cos(\theta) = 2\alpha^2 - 1$  we obtain

$$\text{Var}(Y^{(n)}(s, t)) = \sum_{k=0}^{[ns]-1} \sum_{\ell=0}^{[nt]-1} (\cos(\theta/2))^{2|k-\ell|} \left( P_{k \wedge \ell}^{(0, |k-\ell|)} (\cos(\theta)) \right)^2, \quad (4.4)$$

where  $P_n^{(a,b)}(x)$  is the  $n$ th Jacobi polynomial with parameters  $a$  and  $b$ . Obviously, as  $P_n^{(0,0)}(-1) = (-1)^n$ , in the trivial case  $\alpha = \beta = 0$  ( $\theta = \pi$ ) limit (4.1) can be obtained from (4.4), too. However, as in general the second parameter of the Jacobi polynomial in (4.4) equals  $|k - \ell|$ , to find the limit of the appropriately normed variances of  $Y^{(n)}(s, t)$  the

classical approximations of the Jacobi polynomials as e.g. Theorem 8.21.8 of Szegő (1939) can not be used.

**Acknowledgments.** Research has been supported by the Hungarian Scientific Research Fund under Grant No. OTKA T079128/2009 and partially supported by TÁMOP 4.2.1./B-09/1/KONV-2010-0007/IK/IT project. The project is implemented through the New Hungary Development Plan co-financed by the European Social Fund, and the European Regional Development Fund.

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